

## A double Myers-Perry black hole in five dimensions

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Carlos A. R. Herdeiro, Carmen Rebelo, Miguel Zilhão and Miguel S. Costa

*Depart. de Física e Centro de Física do Porto,*

*Faculdade de Ciências da Univ. do Porto,*

*Rua do Campo Alegre, 687, 4169-007 Porto, Portugal*

*E-mail: crherdei@fc.up.pt, mrebelo@fc.up.pt, c0302009@alunos.fc.up.pt,*

*miguelc@fc.up.pt*

ABSTRACT: Using the inverse scattering method we construct a six-parameter family of exact, stationary, asymptotically flat solutions of the 4+1 dimensional vacuum Einstein equations, with  $U(1)^2$  rotational symmetry. It describes the superposition of two Myers-Perry black holes, each with a *single* angular momentum parameter, both in the same plane. The black holes live in a background geometry which is the Euclidean C-metric with an extra flat time direction. This background possesses conical singularities in two adjacent compact regions, each corresponding to a set of fixed points of one of the  $U(1)$  actions in the Cartan sub-algebra of  $SO(4)$ . We discuss several aspects of the black holes geometry, including the conical singularities arising from force imbalance, and the torsion singularity arising from torque imbalance. The double Myers-Perry solution presented herein is considerably simpler than the four dimensional double Kerr solution and might be of interest in studying spin-spin interactions in five dimensional general relativity.

KEYWORDS: Integrable Equations in Physics, Black Holes in String Theory, Black Holes.

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## 1. Introduction

The paradigmatic example of a static, regular (on and outside an event horizon) multi-black hole spacetime is the family of Majumdar-Papapetrou solutions of Einstein-Maxwell theory in four dimensions [1, 2]. All of the individual objects in these configurations are extremal Reissner-Nordström black holes [3], which are held in equilibrium due to a balance between gravitational attraction and electrostatic repulsion for any pair of black holes. Such force balance is mathematically realised by an *exact* linearisation of the full Einstein-Maxwell equations. This linearisation is most easily obtained by taking the Einstein-Maxwell theory as the bosonic sector of  $\mathcal{N} = 2$ ,  $D = 4$  Supergravity and searching for static, supersymmetric backgrounds with a timelike Killing vector field; the Majumdar-Papapetrou family is the most general such solution [4].

It turns out that the Majumdar-Papapetrou family is not the most general *stationary*, supersymmetric background within  $\mathcal{N} = 2$ ,  $D = 4$  Supergravity, even demanding asymptotic flatness [5]; the most general such solution is the Israel-Wilson-Perjes (IWP) family [6, 7]. For a specific choice, it represents a set of Kerr-Newman “particles” (naked singularities), each of which is obtained by giving spin to an extremal Reissner-Nordström black hole. The force balance is now more involved: in addition to the monopole-monopole

gravitational attraction and electrostatic repulsion, we have dipole-dipole forces. The gravitational one is a spin-spin force, first discussed by Wald [8] using Papapetrou's equation for a spinning particle [9]. Wald showed that, in an appropriate limit, this force has exactly the same form as the usual dipole-dipole force in magnetostatics, but with *opposite sign*.<sup>1</sup> This fact clarifies why there is a force balance in the IWP spacetimes, independently of the orientation of the spin of the individual black holes. This is furthermore confirmed by a probe computation for a charged spinning particle in an IWP spacetime [11] and by a post-post Newtonian analysis of the metric generated by two massive charged spinning sources in the Einstein-Maxwell theory [12]. Note that, since the magnetic dipole of a charged spinning black hole is not an independent quantity, the gyromagnetic ratio plays a crucial role in the cancellation of dipole forces.

But the balance of forces does not guarantee equilibrium in the presence of dipoles. We also have to discuss the balance of *torques*, which is more subtle. Like in magnetostatics, in general relativity non-aligned gravitational dipoles (spinning bodies) also produce a torque on each other [13], which has been recently tested by the Gravity Probe B experiment. This torque obviously vanishes when the two spins are aligned, but *not the total torque*. Imagine that a Schwarzschild black hole is placed in the vicinity of a Kerr black hole, with the spin of the latter parallel to the direction of separation. One could impose a constraint (in the form of a strut) preventing the two black holes from approaching, i.e. from gaining linear acceleration. If no constraint is imposed in the form of a torque, we would expect the Schwarzschild black hole to gain angular acceleration, due to the dragging of inertial frames caused by the Kerr black hole. Thus, in the gravitational case, there seems to be an additional torque, besides the aforementioned one.

If this additional torque is present we might expect some signature in a multi black hole spacetime. Indeed, it was shown in [14] that the rotation one-form in a two (aligned) particles IWP spacetime will diverge somewhere along the axis - either in between the particles or in the remaining of the axis - unless a certain requirement, which we dub *axis condition*, is obeyed (c.f. section 4.2). Failure to obey this condition has been interpreted as a "torsion singularity" in [12, 15, 16]; therein the condition arises as the requirement that the azimuthal vector field has a fixed point at the axis and is spatial otherwise. The analysis in [12] also suggests that, for charged rotating black holes, there is an electromagnetic contribution to the effect that makes a Schwarzschild black hole rotate in the vicinity of a Kerr black hole. However, it so happens that for this effect the purely gravitational and electromagnetic contributions do not completely cancel, even in the supersymmetric case of IWP particles. It is worth noting that the post-post Newtonian analysis suggests that, for uncharged sources, the *regularity condition*, i.e. the requirement of absence of conical singularity representing struts or strings necessary for force balance, is incompatible with the axis condition [15].

The regularity condition has been studied at the level of exact, non-supersymmetric, static solutions in the multi-Schwarzschild [17, 18] and the multi-Schwarzschild-Tangherlini [19]

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<sup>1</sup>Actually, using a gravito-electromagnetic analogy based on tidal tensors [10], the Papapetrou equation for a spinning particle can be simply derived from the force acting on a magnetic dipole in magnetostatics.

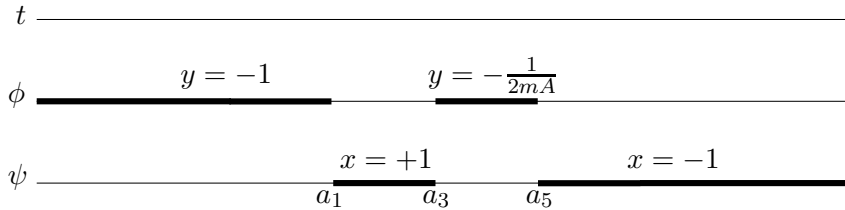
spacetimes. However, to study the regularity and axis condition at the level of exact, non-supersymmetric solutions seems, in principle, a much more difficult task, mainly because such solutions, which are stationary rather than static, are usually rather involved. The paradigmatic example is the double-Kerr solution, originally generated in [20] via a Bäcklund transformation. The complexity of this solution has led to different claims concerning the explicit form of these conditions (see, for instance [21, 16, 22]), although it is a consensual conclusion that the solution for two black holes must have singularities, in agreement with the spinning test particle analysis of [8]. It turns out that a five dimensional version of the double-Kerr solution, the *double Myers-Perry* solution, is drastically simpler than its four dimensional counterpart. The reason is simply understood: using the Belinskii-Zakharov inverse scattering method [23, 24], the Kerr solution [25] is generated by a 2-soliton transformation, whereas the Myers-Perry solution [26] with a single angular momentum can be generated by a *single* soliton transformation (see [27] for a review of the inverse scattering method and applications). Thus, whereas the double-Kerr solution is generated by a 4-soliton transformation [16], the double-Myers-Perry solution is generated, effectively, by a 2-soliton transformation.<sup>2</sup> To generate the latter solution is the main purpose of the present paper. This will allow us to write down in a very simple and clear fashion the regularity and axis conditions for this spacetime.

The new solution presented herein is also of interest in a different context. Over the last few years a great effort has been made to tackle the black hole classification problem in higher dimensions [27]. It is well known that the “phase space” of regular (i.e. free of curvature singularities on and outside an event horizon) and asymptotically flat black objects is rather richer than in four dimensions, containing exotic objects such as black rings [28–34] and black saturns [35]; equivalently there are no (simple) black hole uniqueness theorems analogous to the four dimensional case for vacuum, stationary configurations. The new stationary solution presented herein describes the superposition of two Myers-Perry black holes in five dimensions, each with a single angular momentum parameter, both in the same plane. The black holes live in a background geometry, which is the Euclidean C-metric with an extra flat time direction. The downside of the new solution is that it is built upon a non-trivial background geometry with conical singularities, which are still present, generically, when the black holes are included. It remains to be seen if, by including the second angular momentum parameter or other fields, like the electromagnetic field, such singularities can be removed.

This paper is organised as follows. In section two we analyse the background geometry upon which the double Myers-Perry solution will be built. The use of this background is actually a necessity for using the inverse scattering method. In section 3 we discuss the static solution, first constructed in [19], that will be used as the seed metric for the new solution presented herein. In section 4 the double Myers-Perry solution is generated using the inverse scattering method and its rod structure is analysed; other basic properties of the solution as well as the computation of the relevant physical quantities are presented in

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<sup>2</sup>Actually we will use a 4-soliton transformation, but since two of the solitons will have trivial BZ vectors, it is effectively as complex as a 2-soliton transformation.



**Figure 1:** Rod structure for the background spacetime. Next to each rod we write its locus in  $xy$  coordinates. The rods correspond to the edges of the rectangle in figure 2 (right). In terms of the parameters  $m$  and  $A$ , the  $a_i$ 's can be taken as  $a_1 = -\frac{1}{2A^2}$ ,  $a_3 = -\frac{m}{A}$ ,  $a_5 = \frac{m}{A}$ .

section 5. We close with a discussion.

## 2. Background geometry: the Euclidean C-metric

In principle one could have a double Myers-Perry solution in five dimensions that would reduce to flat space when the two black holes are removed. Such solution, however, could not have a  $U(1)^2$  spatial isometry which, together with time translations, yields the three commuting Killing vector fields necessary to apply the inverse scattering method that we shall use to generate the new solution. Indeed, placing two separated point-like sources in five dimensional Minkowski spacetime reduces the spatial isometry to  $SO(3)$ . Introducing rotation breaks this symmetry group further; at most we end up with  $SO(2)$ . Thus, such solution could not be generated by the Weyl or inverse scattering techniques and such problem seems very difficult to approach [36].

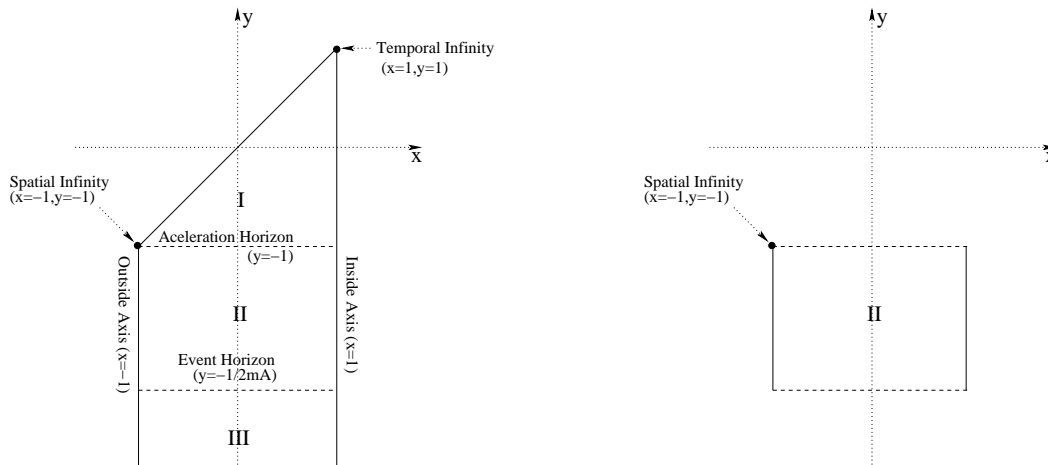
To generate a solution with two black holes (with topologically  $S^3$  horizons) with the Weyl and inverse scattering techniques, we need a  $U(1)^2$  spatial isometry and hence a background with at least *two* fixed points of the two  $U(1)$  actions. Flat space has only one such point, as it is clear from its rod structure. A background with two such points would be the four dimensional Euclidean Schwarzschild with an added time direction. However, as it is clear from its rod structure, this background is not asymptotically flat. The black holes one can superimpose on this background live on Kaluza-Klein bubbles, and they have been constructed in [37–39].

In order to have at least two fixed points and asymptotic flatness we need a background with three fixed points, which is exactly what happens for the Euclidean C-metric with an extra flat time direction; thus this geometry is our background in the absence of the two black holes. Its rod structure is represented in figure 1.

The *Lorentzian* C-metric can be written in  $xy$  coordinates as [40]

$$ds^2 = \frac{1}{A^2(x-y)^2} \left[ G(y)dt^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + G(x)d\tilde{\psi}^2 \right], \quad G(\xi) \equiv (1 - \xi^2)(1 + 2mA\xi),$$

with  $0 < 2mA < 1$ . The coordinate range for the  $xy$  coordinates is displayed in figure 2 (left). For the Lorentzian solution, these are  $-1 \leq x \leq 1$  for  $-\infty < y \leq -1$  and  $-1 \leq y < x \leq 1$ . The latter (region I) corresponds to a “Milne” region wherein the coordinate  $t$  is spacelike and  $y$  is timelike, a behaviour also found for these coordinates when  $y < -1/2mA$  (region III).



**Figure 2:**  $xy$  coordinate space for the Lorentzian (left) and Euclidean (right) C-metric. For the latter each edge is a fixed point set of a given periodic vector field.

The Euclidean C-metric we want to consider is obtained by the analytic continuation  $t \rightarrow i\tilde{\phi}$ . The coordinate range for the  $xy$  coordinates is also displayed in figure 2 (right), corresponding to the rectangular region wherein the  $t$  coordinate was timelike in the Lorentzian regime (region II). The boundary of this region is a set of fixed points of either  $\partial/\partial\tilde{\psi}$  at  $x = \pm 1$ ,  $\partial/\partial\tilde{\phi}$  at  $y = -1, -\frac{1}{2mA}$ , or both (the three vertexes of the rectangle denoted  $a_1, a_3$  and  $a_5$  are *double fixed points*). This can be clearly seen by changing from  $(x, y)$  coordinates to canonical Weyl coordinates  $(\rho, z)$ ; in particular we have

$$\rho^2 = \frac{(y^2 - 1)(1 - x^2)(1 + 2mAy)(1 + 2MAx)}{A^4(x - y)^4(1 - 2mA)^4}.$$

Note that the vertexes of the rectangular region in  $xy$  coordinate space correspond to the breaks in the rod structure of the Euclidean C-metric - figure 1.

There are conical singularities in this background geometry. This is the price to pay to have three double fixed points. We can, however, make the geometry free of conical singularities at spatial infinity. Defining new angular coordinates  $(\phi, \psi) \equiv (1 - 2mA)(\tilde{\phi}, \tilde{\psi})$ , and taking the canonical periodicities  $\Delta\phi = 2\pi = \Delta\psi$ , the edges  $x = -1$  and  $y = -1$  become free of conical singularities. Thus the background geometry becomes asymptotically flat. In the remaining two edges there are conical *excesses* - figure 3 - given by

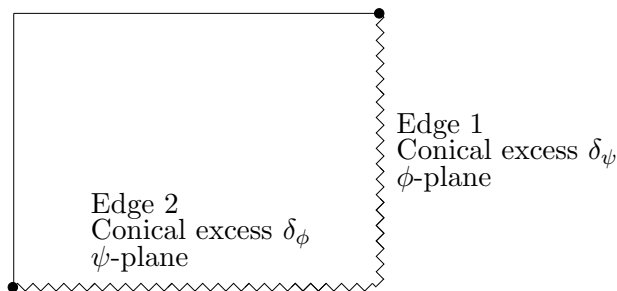
$$\delta_\psi = 2\pi \frac{4mA}{1 - 2mA} = 2\pi \frac{a_{53}}{a_{31}}, \quad x = +1 \Leftrightarrow a_1 < z < a_3, \quad (2.1)$$

$$\delta_\phi = 2\pi \frac{1 - 2mA}{4mA} = 2\pi \frac{a_{31}}{a_{53}}, \quad y = -\frac{1}{2mA} \Leftrightarrow a_3 < z < a_5, \quad (2.2)$$

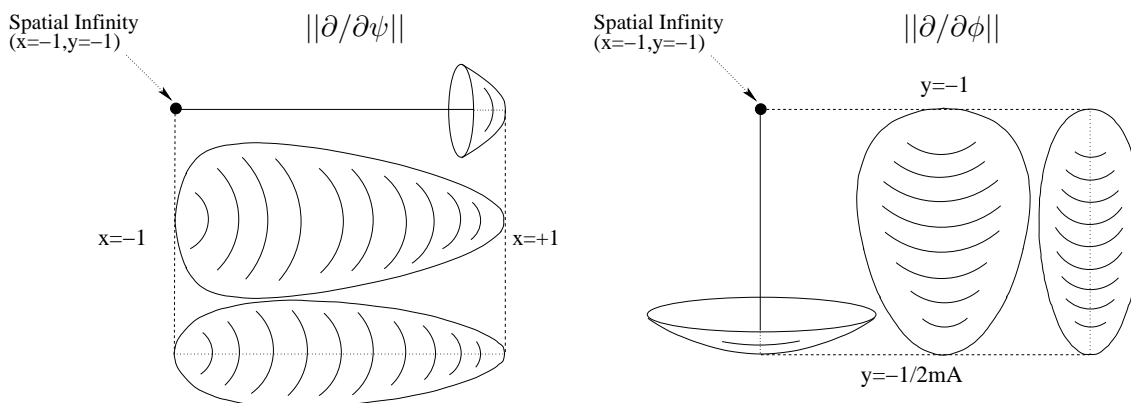
where throughout this paper we use the notation

$$a_{ij} \equiv a_i - a_j.$$

In figure 4 we represent the norm of  $\partial/\partial\psi$  and  $\partial/\partial\phi$  in  $xy$  coordinate space. This gives an idea of the four dimensional Euclidean geometry. In particular, neglecting the conical



**Figure 3:** Conical singularities in  $xy$  coordinate space. After an appropriate choice of  $\phi, \psi$  angles, with canonical period  $2\pi$  the only conical singularities are found at the edges  $x = +1$  (Edge 1),  $y = -\frac{1}{2mA}$  (Edge 2). The two double fixed points where black holes will be placed are also emphasised.



**Figure 4:**  $xy$  coordinate space; Left: Norm of  $\frac{\partial}{\partial\psi}$  along  $y = -\frac{1}{2mA}, -\frac{1}{mA}, -1$ ; note that  $x = \pm 1$  are fixed point sets of this vector field, but with our choice of angular coordinate there are conical singularities only at  $x = +1$ ; Right: Norm of  $\frac{\partial}{\partial\phi}$  along  $x = -1, 0, +1$ ; note that  $y = -1, -\frac{1}{2mA}$  are fixed point sets of this vector field, but with our choice of angular coordinate there are conical singularities only at  $y = -\frac{1}{2mA}$ .

singularities, its topology is  $S^2 \times S^2 - \{P\}$ , where the point  $P$  corresponds to spatial infinity wherein these norms diverge. This topology is analogous to that of the instanton considered in [41].

In terms of canonical Weyl coordinates, the metric of the five dimensional background geometry has the form<sup>3</sup>

$$ds^2 = -dt^2 + \frac{\mu_3}{\mu_1\mu_5}\rho^2 d\phi^2 + \frac{\mu_1\mu_5}{\mu_3} \left( d\psi^2 + \frac{(\rho^2 + \mu_1\mu_3)^2(\rho^2 + \mu_3\mu_5)^2 [d\rho^2 + dz^2]}{(\rho^2 + \mu_1\mu_5)^2(\rho^2 + \mu_1^2)(\rho^2 + \mu_3^2)(\rho^2 + \mu_5^2)} \right), \tag{2.3}$$

where

$$\mu_k \equiv \sqrt{\rho^2 + (z - a_k)^2} - (z - a_k).$$

This metric is invariant under the exchange

$$a_1 \leftrightarrow a_5. \tag{2.4}$$

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<sup>3</sup>Note that the dimensions of these coordinates are  $[\rho] = [z] = L^2$ .

However notice that the conical excesses in the  $\phi$  and  $\psi$  plane are interchanged. Since a generalisation of this invariance will hold in the presence of the two black holes, let us comment on it. The physical information that determines the geometry is given by the sizes of the two finite rods in figure 1. Thus, one of the three parameters that describe the geometry  $(a_1, a_3, a_5)$  is redundant. Such redundancy can be gauged away by introducing a new coordinate  $\tilde{z} = z - a_3$ , in terms of which the metric reads

$$ds^2 = -dt^2 + \frac{\mu}{\mu_{13}\mu_{53}}\rho^2 d\phi^2 + \frac{\mu_{13}\mu_{53}}{\mu} \left( d\psi^2 + \frac{(\rho^2 + \mu\mu_{13})^2(\rho^2 + \mu\mu_{53})^2 [d\rho^2 + d\tilde{z}^2]}{(\rho^2 + \mu_{13}\mu_{53})^2(\rho^2 + \mu_{13}^2)(\rho^2 + \mu^2)(\rho^2 + \mu_{53}^2)} \right),$$

where

$$\mu_{k3} \equiv \sqrt{\rho^2 + (\tilde{z} - a_{k3})^2} - (\tilde{z} - a_{k3}), \quad \mu = \mu_{33}.$$

Fixing the physical information, i.e. the rod sizes  $a_{53}$  and  $a_{31}$ , it is simple to show that the metric is invariant under

$$(a_{31}, a_{53}; \tilde{z}, \psi, \phi) \rightarrow (a_{53}, a_{31}; -\tilde{z}, \phi, \psi).$$

This follows easily by noting that under this transformation

$$\mu_{53} \rightarrow \frac{\rho^2}{\mu_{13}}, \quad \mu_{13} \rightarrow \frac{\rho^2}{\mu_{53}}, \quad \mu \rightarrow \frac{\rho^2}{\mu}.$$

This is nothing but the usual invariance of a system of particles on a line under the inversion of the order together with a parity transformation and it is what the transformation (2.4) effectively implements. Noting this invariance will be useful for checking our solution and also for checking physical quantities that describe the whole spacetime. Note that, in the particular case  $a_{31} = a_{53}$ , the background geometry is invariant under  $(\tilde{z}, \psi, \phi) \rightarrow (-\tilde{z}, \phi, \psi)$ ; this is the five dimensional version of the  $\mathbb{Z}_2$  symmetry of, for instance, the equal mass double-Schwarzschild solution.

### 3. The static case: double Schwarzschild-Tangherlini

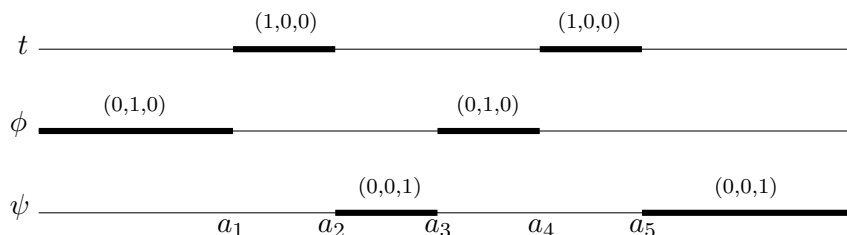
The starting point for the new solution which will be presented in the next section is the double Schwarzschild-Tangherlini spacetime built in [19], using the technique developed in [42], whose rod structure is given in figure 5. We have placed the two timelike rods representing black hole horizons at  $z = a_1, a_5$  in figure 1, so that the conical singularities represented in figure 3 are in between the two black holes. In this way we expect that the interactions between the two black holes might alter significantly these singularities. Note that, throughout this paper, we choose the ordering:

$$a_1 < a_2 < a_3 < a_4 < a_5. \quad (3.1)$$

In the static case the metric is essentially read off from the rod structure:

$$ds^2 = -\frac{\mu_1\mu_4}{\mu_2\mu_5}dt^2 + \frac{\mu_3}{\mu_1\mu_4}\rho^2 d\phi^2 + \frac{\mu_2\mu_5}{\mu_3}d\psi^2 + k \frac{\mu_2\mu_5}{\mu_3} \frac{\prod_{i<j}(\rho^2 + \mu_i\mu_j) [d\rho^2 + dz^2]}{(\rho^2 + \mu_1\mu_4)^3(\rho^2 + \mu_2\mu_5)^3 \prod_{i=1}^5(\rho^2 + \mu_i^2)}, \quad (3.2)$$





**Figure 5:** Rod structure for the double Schwarzschild-Tangherlini. Next to each rod the corresponding eigenvector [43] is displayed.

where  $k$  is an integration constant. One can verify that the metric is invariant under the exchange

$$(a_1, a_2) \leftrightarrow (a_4, a_5), \quad (3.3)$$

which generalises (2.4) in the presence of static black holes. Taking  $k = 1$  and the periodicities  $\Delta\phi = 2\pi = \Delta\psi$  guarantees this solution is asymptotically flat. There are, however, conical singularities for  $a_2 < z < a_3$  and  $a_3 < z < a_4$  in the  $\rho - \psi$  and  $\rho - \phi$  planes, respectively. The conical *excesses* are, respectively [19]

$$\begin{aligned} \delta_\psi &= 2\pi \left( \frac{a_{41}a_{52}}{\sqrt{a_{51}a_{31}a_{32}a_{42}}} - 1 \right), \quad a_2 \leq z < a_3; \\ \delta_\phi &= 2\pi \left( \frac{a_{41}a_{52}}{\sqrt{a_{51}a_{43}a_{53}a_{42}}} - 1 \right), \quad a_3 < z \leq a_4. \end{aligned} \quad (3.4)$$

It is straightforward to show that, for the ordering (3.1),  $\delta_\psi$  and  $\delta_\phi$  are strictly positive. Hence *there is no choice of parameters that makes the background free of conical singularities in the static case*. We will contrast this state of affairs with that of the stationary solution presented below.

The mass, area and temperature of each black hole can be written as (we set the five dimensional Newton constant to one)

$$M_1^{\text{Komar}} = \frac{3\pi}{8} \Delta, \quad \mathcal{A}_1 = 2\pi^2 \frac{\sqrt{2a_{21}a_{31}a_{51}}}{a_{41}} \Delta, \quad T_1 = \frac{1}{2\pi} \sqrt{\frac{2a_{21}}{a_{31}a_{51}}} \frac{a_{41}}{\Delta}, \quad (3.5)$$

$$M_2^{\text{Komar}} = \frac{3\pi}{8} \bar{\Delta}, \quad \mathcal{A}_2 = 2\pi^2 \frac{\sqrt{2a_{54}a_{53}a_{51}}}{a_{52}} \bar{\Delta}, \quad T_2 = \frac{1}{2\pi} \sqrt{\frac{2a_{54}}{a_{53}a_{51}}} \frac{a_{52}}{\bar{\Delta}}, \quad (3.6)$$

where the individual masses can be computed as Komar integrals at each horizon (cf. section 5.5), and for reasons that will become clear later we introduced

$$\Delta \equiv 2a_{21}, \quad \bar{\Delta} \equiv 2a_{54}. \quad (3.7)$$

These quantities are consistent with the Smarr-type formula

$$\frac{2}{3} M_i^{\text{Komar}} = T_i \frac{\mathcal{A}_i}{4}, \quad i = 1, 2. \quad (3.8)$$

Note also that in this case the two black hole masses add up to the ADM mass of the spacetime:

$$M_{\text{ADM}} = M_1^{\text{Komar}} + M_2^{\text{Komar}}. \quad (3.9)$$

Finally, observe that under (3.3) the physical masses and conical excesses are interchanged, as one would expect:

$$M_1^{\text{Komar}} \leftrightarrow M_2^{\text{Komar}}, \quad \delta_\psi \leftrightarrow \delta_\phi.$$

## 4. The stationary case: double Myers-Perry

### 4.1 Generating the solution with the inverse scattering method

In  $D$  spacetime dimensions, the inverse scattering method (or Belinskii-Zakharov method) [23, 24] can be used to construct new Ricci flat metrics with  $D - 2$  commuting Killing vector fields from known ones, by using purely algebraic manipulations. Such metrics can always be written in the form

$$ds^2 = G_{ab}(\rho, z) dx^a dx^b + e^{2\nu(\rho, z)}(d\rho^2 + dz^2), \quad (4.1)$$

where  $a, b = 1, \dots, D - 2$ . In what follows we shall specialise all results of the method to the case of interest herein; in particular  $D = 5$ .

The seed metric is the double Schwarzschild-Tangherlini spacetime (3.2):

$$G_0 = \text{diag} \left\{ -\frac{\mu_1\mu_4}{\mu_2\mu_5}, -\frac{\bar{\mu}_1\mu_3}{\mu_4}, \frac{\mu_2\mu_5}{\mu_3} \right\}. \quad (4.2)$$

As usual the  $\mu$ 's refer to soliton positions in the BZ method and

$$\tilde{\mu}_k = \pm \sqrt{\rho^2 + (z - a_k)^2} - (z - a_k);$$

the “+” pole refers to a *soliton* and is denoted by  $\mu_k$ ; the “−” pole refers to an *anti-soliton* and is denoted by  $\bar{\mu}_k$ . For the seed solution the conformal factor is, from (3.2),

$$e^{2\nu_0} = k \frac{\mu_2\mu_5}{\mu_3} \frac{\prod_{i < j} (\rho^2 + \mu_i\mu_j)}{(\rho^2 + \mu_1\mu_4)^3 (\rho^2 + \mu_2\mu_5)^3 \prod_{i=1}^5 (\rho^2 + \mu_i^2)}, \quad (4.3)$$

where  $k$  is an integration constant.

We proceed with the method suggested by Pomersky [44] (see also [27] for a recent review) and implement the following 4-soliton transformation: we remove two anti-solitons, at  $z = a_1$  and  $z = a_4$ , and two solitons, at  $z = a_2$  and  $z = a_5$ , all with BZ vectors  $(1, 0, 0)$ . Thus we divide  $(g_0)_{tt}$  by  $\rho^8 / \bar{\mu}_1^2 \bar{\mu}_4^2 \mu_2^2 \mu_5^2$ . The seed metric becomes

$$G'_0 = \frac{\mu_2\mu_5}{\mu_1\mu_4} \text{diag} \left\{ -1, \frac{\mu_3\rho^2}{\mu_2\mu_5}, \frac{\mu_1\mu_4}{\mu_3} \right\} \equiv \frac{\mu_2\mu_5}{\mu_1\mu_4} \tilde{G}_0. \quad (4.4)$$

We will actually take the rescaled metric  $\tilde{G}_0$  to be our seed (bearing in mind that one should multiply the final metric by the overall factor  $\mu_2\mu_5/\mu_1\mu_4$ ). We take the generating matrix to be

$$\tilde{\Psi}_0(\lambda, \rho, z) = \text{diag} \left\{ -1, -\frac{(\bar{\mu}_2 - \lambda)(\bar{\mu}_5 - \lambda)}{(\bar{\mu}_3 - \lambda)}, \frac{(\mu_1 - \lambda)(\mu_4 - \lambda)}{(\mu_3 - \lambda)} \right\}. \quad (4.5)$$

One can verify that this matrix solves the Lax pair constructed in the BZ method (see [23, 24]). The double Myers-Perry solution is now obtained by a 4-soliton transformation: using  $\tilde{G}_0$  as seed, we add two anti-solitons, at  $z = a_1$  with BZ vector  $m_{0b}^{(1)} = (1, b, 0)$  and at  $z = a_4$  with a BZ vector  $m_{0b}^{(4)} = (1, c, 0)$ , and add two trivial solitons, at  $z = a_2$  with BZ vector  $m_{0b}^{(2)} = (1, 0, 0)$  and at  $z = a_5$  with BZ vector  $m_{0b}^{(5)} = (1, 0, 0)$ .<sup>4</sup> Notice that we have introduced two new parameters:  $b$  and  $c$ . The resulting metric is

$$G = \frac{\mu_2\mu_5}{\mu_1\mu_4} \tilde{G},$$

where  $\tilde{G}$  has components

$$\tilde{G}_{ab} = (\tilde{G}_0)_{ab} - \sum_{k,l} \frac{(\tilde{G}_0)_{ac} m_c^{(k)} (\tilde{\Gamma}^{-1})_{kl} m_d^{(l)} (\tilde{G}_0)_{db}}{\tilde{\mu}_k \tilde{\mu}_l}, \quad (4.6)$$

with  $k, l = 1, 2, 4, 5$  and  $\tilde{\mu}_k = \mu_k$  for  $k = 2, 4$  whereas  $\tilde{\mu}_k = \bar{\mu}_k$  for  $k = 1, 3$ . The space-time components of the four vectors  $m^{(k)}$  are given by

$$m_a^{(k)} = m_{0b}^{(k)} \left[ \tilde{\Psi}_0^{-1}(\tilde{\mu}_k, \rho, z) \right]_{ba}. \quad (4.7)$$

The symmetric matrix  $\tilde{\Gamma}$ , whose inverse is  $\tilde{\Gamma}^{-1}$ , reads

$$\tilde{\Gamma}_{kl} = \frac{m_a^{(k)} (\tilde{G}_0)_{ab} m_b^{(l)}}{\rho^2 + \tilde{\mu}_k \tilde{\mu}_l}. \quad (4.8)$$

Finally, it only remains to compute the function  $\nu$  in the metric, which is given by

$$e^{2\nu} = e^{2\nu_0} \frac{\det \Gamma_{kl}}{\det \Gamma_{kl}^{(0)}}, \quad (4.9)$$

where  $\Gamma^{(0)}$  and  $\Gamma$  are constructed as in (4.8) using  $G_0$  and  $G$ , respectively.

The end result of the above algorithm can be written in the following form, analogous to the black saturn solution [35]

$$ds^2 = -\frac{H_y}{H_x} \left[ dt + \left( \frac{\omega_\phi}{H_y} - q \right) d\phi \right]^2 + \frac{H_x}{H_y} \frac{\rho^2 \mu_3}{\mu_2 \mu_5} d\phi^2 + \frac{\mu_2 \mu_5}{\mu_3} d\psi^2 + k \frac{H_x}{F} (d\rho^2 + dz^2), \quad (4.10)$$

---

<sup>4</sup>This 4-soliton transformation allows us to work with the simplest possible seed (rescaled metric  $\tilde{G}_0$ ). Moreover, it would now be straightforward, even if computationally challenging, to generate the general double doubly spinning Myers Perry (i.e the solution where each black hole has angular momentum in both planes) just by considering the non trivial BZ vectors:  $m_{0b}^{(1)} = (1, b, 0)$ ,  $m_{0b}^{(2)} = (1, 0, d)$ ,  $m_{0b}^{(4)} = (1, c, 0)$  and  $m_{0b}^{(5)} = (1, 0, e)$ .

where a coordinate transformation  $dt \rightarrow dt - q d\phi$  was performed;  $q$  will be chosen below. The metric functions are<sup>5</sup>

$$\begin{aligned} H_x &= M_0 + b^2 M_1 + c^2 M_2 + bc M_3 + b^2 c^2 M_4, \\ H_y &= \frac{\rho^2}{\mu_2 \mu_5} \left[ M_0 \frac{\mu_1 \mu_4}{\rho^2} - b^2 M_1 \frac{\mu_4}{\mu_1} - c^2 M_2 \frac{\mu_1}{\mu_4} - bc M_3 + b^2 c^2 M_4 \frac{\rho^2}{\mu_1 \mu_4} \right], \\ \omega_\phi &= 2 \sqrt{\frac{\mu_3}{\mu_2 \mu_5}} \left[ b R_1 \sqrt{M_0 M_1} + c R_4 \sqrt{M_0 M_2} - b^2 c R_4 \sqrt{M_1 M_4} - bc^2 R_1 \sqrt{M_2 M_4} \right]; \end{aligned} \quad (4.11)$$

where  $R_i = \sqrt{\rho^2 + (z - a_i)^2}$  and the functions  $M_i$  are

$$\begin{aligned} M_0 &\equiv \mu_2 \mu_3^2 \mu_5 (\mu_1 - \mu_4)^2 (\rho^2 + \mu_1 \mu_2)^2 (\rho^2 + \mu_1 \mu_5)^2 (\rho^2 + \mu_2 \mu_4)^2 (\rho^2 + \mu_4 \mu_5)^2, \\ M_1 &\equiv \mu_1^2 \mu_2^2 \mu_3 \mu_5^2 (\mu_1 - \mu_3)^2 (\rho^2 + \mu_1 \mu_4)^2 (\rho^2 + \mu_2 \mu_4)^2 (\rho^2 + \mu_4 \mu_5)^2, \\ M_2 &\equiv \mu_2^2 \mu_3 \mu_4^2 \mu_5^2 (\mu_3 - \mu_4)^2 (\rho^2 + \mu_1 \mu_2)^2 (\rho^2 + \mu_1 \mu_4)^2 (\rho^2 + \mu_1 \mu_5)^2, \\ M_3 &\equiv 2 \mu_1 \mu_2^2 \mu_3 \mu_4 \mu_5^2 (\mu_1 - \mu_3) (\mu_3 - \mu_4) (\rho^2 + \mu_1^2) (\rho^2 + \mu_4^2) (\rho^2 + \mu_1 \mu_2) (\rho^2 + \mu_1 \mu_5) \\ &\quad \times (\rho^2 + \mu_2 \mu_4) (\rho^2 + \mu_4 \mu_5), \\ M_4 &\equiv \mu_1^2 \mu_2^3 \mu_4^2 \mu_5^3 \rho^4 (\mu_1 - \mu_3)^2 (\mu_1 - \mu_4)^2 (\mu_3 - \mu_4)^2. \end{aligned} \quad (4.12)$$

Moreover

$$F = \mu_3^3 (\mu_1 - \mu_4)^2 (\rho^2 + \mu_1 \mu_2) (\rho^2 + \mu_1 \mu_4)^2 (\rho^2 + \mu_1 \mu_5) (\rho^2 + \mu_2 \mu_5)^2 (\rho^2 + \mu_2 \mu_4) \times (\rho^2 + \mu_4 \mu_5) \prod_{i=1}^5 (\rho^2 + \mu_i^2) / [(\rho^2 + \mu_1 \mu_3) (\rho^2 + \mu_2 \mu_3) (\rho^2 + \mu_3 \mu_4) (\rho^2 + \mu_3 \mu_5)].$$

The metric (4.10) is invariant under the exchange

$$(a_1, a_2, b) \leftrightarrow (a_4, a_5, c), \quad (4.13)$$

which generalises (2.4) and (3.3) to the case of two stationary black holes.

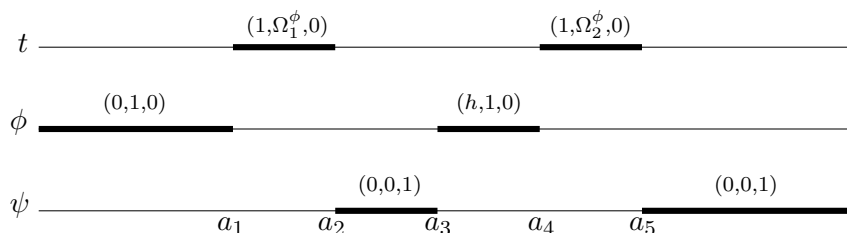
Let us note that, despite the high degree of complexity of this solution, it is drastically simpler than the four dimensional double Kerr solution, originally obtained via a Bäcklund transformation [20]. From the viewpoint of the inverse scattering method this can be understood from the fact that the double Kerr can only be constructed, from a double Schwarzschild seed, by a four-soliton transformation with *all* solitons having non-trivial BZ vectors.

## 4.2 Rod structure, horizons angular velocities and axis condition

The rod structure of the solution we have just generated is the same as the one of the static solution, except for the directions of the rods - figure 6. From this rod structure it is clear that the metric gives a six parameter family of solutions. The parameters can be taken to be the four finite rod sizes, together with  $b$  and  $c$ . Physically, the six independent degrees of freedom can be taken to be the two black hole masses and angular momenta,

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<sup>5</sup>Following standard notation, the square roots of the function  $M_i$  are to be understood as, for example,  $\sqrt{(\mu_1 - \mu_4)^2} = \mu_1 - \mu_4$ .



**Figure 6:** Rod structure for the double Myers-Perry spacetime. Next to each rod the corresponding eigenvector [43] is displayed.

together with the two conical singularities. Alternatively one can replace the two conical singularities by the two distances  $d_1 \equiv a_{32}$  and  $d_2 \equiv a_{43}$ . Note that  $d = d_1 + d_2$  is the (coordinate) distance squared between the two black holes.

The eigenvector of the two timelike rods gains a spatial component, along the  $\phi$  direction. These new components are the angular velocities of the individual black hole horizons. A computation shows that they take the form

$$\Omega_1^\phi = \frac{a_{41}b}{a_{51}\Delta}, \quad \Omega_2^\phi = \frac{a_{54}\tilde{b} + a_{51}\tilde{c}}{a_{41}a_{51}\bar{\Delta}}, \quad (4.14)$$

where, for convenience, we have introduced the quantities

$$\Delta \equiv 2a_{21} + \frac{a_{31}b^2}{a_{51}}, \quad \bar{\Delta} \equiv 2a_{54} + \frac{(\tilde{b} + \tilde{c})(a_{54}\tilde{b} + a_{51}\tilde{c})}{a_{41}^2 a_{51}}, \quad (4.15)$$

which generalise (3.7) for the stationary case, and

$$\tilde{b} \equiv a_{31}b, \quad \tilde{c} \equiv a_{43}c. \quad (4.16)$$

These angular velocities reduce to the horizon angular velocities of single Myers-Perry black holes in the limits  $a_3 = a_4 = a_5$  and  $a_1 = a_2 = a_3$ , respectively (cf. (5.1) and (5.2)).

The finite rod between  $a_3$  and  $a_4$  (figure 6) also gains a timelike component,

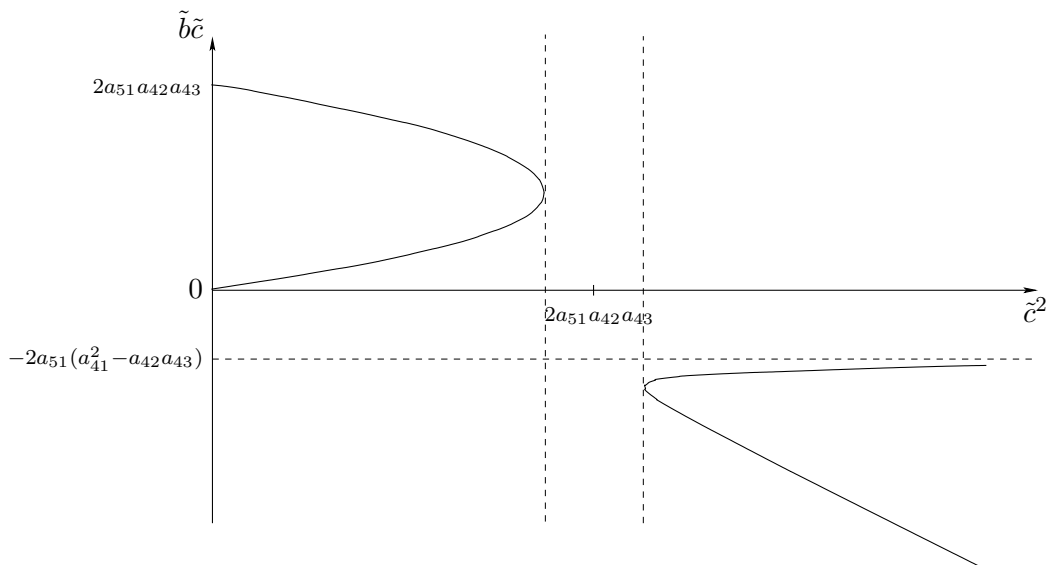
$$h = - \left( \frac{g_{\phi\phi}}{g_{t\phi}} \right)_{\rho=0, a_3 < z < a_4} = \frac{(\tilde{b} + \tilde{c})(2a_{42}a_{51} - \tilde{b}\tilde{c}) - 2a_{41}^2 a_{51}c}{a_{41}(2a_{42}a_{51} - \tilde{b}\tilde{c})}.$$

Thus  $h = 0$  iff  $(g_{\phi\phi})_{\rho=0, a_3 < z < a_4} = 0$ . The latter is sometimes called the *axis condition* [15] (see also [16]); if violated,  $\rho = 0$  and  $a_3 < z < a_4$  is not an axis for  $\partial/\partial\phi$ ; moreover, if  $h \neq 0$  there are naked closed timelike curves in spacetime for some choices of  $b$  and  $c$ , which are generically regarded as pathological. Thus, we demand  $h = 0$ , which yields the constraint

$$\Delta_{\text{axis}} = 0, \quad \Delta_{\text{axis}} \equiv (\tilde{b} + \tilde{c})(2a_{42}a_{51} - \tilde{b}\tilde{c}) - 2a_{41}^2 a_{51}c. \quad (4.17)$$

In particular, this equation is obeyed if  $b = 0 = c$ , as expected. It is also obeyed if we take the limit in which the first black hole disappears, i.e.  $a_1 = a_2 = a_3$ . Note that it does not make sense to consider the limit of (4.17) in which the second black hole disappears, i.e.  $a_3 = a_4 = a_5$ , since in that limit the rod whose direction defines the axis condition collapses to zero size. In general, (4.17) can be regarded as an equation defining  $\tilde{c}^2$  in terms of  $\tilde{b}\tilde{c}$ :

$$\tilde{c}^2 = \frac{(2a_{42}a_{43}a_{51} - \tilde{b}\tilde{c})\tilde{b}\tilde{c}}{\tilde{b}\tilde{c} + 2a_{51}(a_{41}^2 - a_{42}a_{43})}. \quad (4.18)$$



**Figure 7:** Axis condition (4.17):  $\tilde{b}\tilde{c}$  can only take the values (4.19).

Positivity of the left hand side restricts the possible values of  $\tilde{b}\tilde{c}$  to

$$-\infty < \tilde{b}\tilde{c} < -2a_{51}(a_{41}^2 - a_{42}a_{43}) \quad \vee \quad 0 < \tilde{b}\tilde{c} < 2a_{51}a_{42}a_{43}, \quad (4.19)$$

as displayed in figure 7.

## 5. Analysis of the double Myers-Perry solution

### 5.1 Single black hole limits

Let us now see that the solution (4.10) indeed contains two Myers-Perry black holes. First collapse the rod structure of the second black hole by taking  $a_3 = a_4 = a_5$ . To establish that the resulting metric describes a Myers-Perry black hole with a single angular momentum parameter it is convenient to change from Weyl canonical coordinates  $(\rho, z)$  to prolate spherical coordinates  $(x, y)$  by

$$\mu_{1,2} = \alpha(x \mp 1)(1 - y), \quad 2\alpha \equiv a_{21},$$

so that  $\rho^2 = \alpha^2(x^2 - 1)(1 - y^2)$ . Defining also  $\rho_0^2 \equiv 4\alpha + b^2$ , the metric coefficients become (take  $q = b$  so that  $g_{t\phi} \rightarrow 0$  asymptotically)

$$\begin{aligned} G_{tt} &= -\frac{4\alpha x - b^2 y - \rho_0^2}{4\alpha x - b^2 y + \rho_0^2}, & G_{t\phi} &= -\frac{b\rho_0^2(1+y)}{4\alpha x - b^2 y + \rho_0^2}, \\ G_{\phi\phi} &= \frac{1+y}{4} \left( 4\alpha x + b^2 + \rho_0^2 + \frac{2b^2\rho_0^2(1+y)}{4\alpha x - b^2 y + \rho_0^2} \right), \\ G_{\psi\psi} &= \alpha(1-y)(1+x), & e^{2\nu} &= k \frac{H_x}{F} = \frac{4\alpha x - b^2 y + \rho_0^2}{8\alpha^2(x^2 - y^2)}, \end{aligned}$$

where we have taken  $k = 1$  and a standard  $2\pi$  period for the azimuthal angles; this choices make the geometry free of conical singularities. The above metric coefficients coincide

with those of the Myers-Perry black hole with one angular momentum [43]. The ADM mass (which equals the Komar mass), ADM angular momentum (which equals the Komar angular momentum), horizon angular velocity, area and temperature of this black hole are given, respectively, by

$$M_1^{\text{Komar}} = \frac{3\pi}{8} \Delta_1, \quad J_1^\phi = \frac{\pi}{4} b \Delta_1, \quad \Omega_1^\phi = \frac{b}{\Delta_1}, \quad \mathcal{A}_1 = 2\pi^2 \sqrt{2a_{21}} \Delta_1, \quad T_1 = \frac{1}{2\pi} \frac{\sqrt{2a_{21}}}{\Delta_1}, \quad (5.1)$$

where

$$\Delta_1 \equiv 2a_{21} + b^2.$$

Similarly we can collapse the rod structure of the first black hole by taking  $a_1 = a_2 = a_3$ . All of the above steps can be repeated, with the replacements  $a_{21} \rightarrow a_{54}$  and  $b \rightarrow c$ . One finds another Myers-Perry black hole, with ADM mass, ADM angular momentum, horizon angular velocity, area and temperature given by

$$M_2^{\text{Komar}} = \frac{3\pi}{8} \Delta_2, \quad J_2^\phi = \frac{\pi}{4} c \Delta_2, \quad \Omega_2^\phi = \frac{c}{\Delta_2}, \quad \mathcal{A}_2 = 2\pi^2 \sqrt{2a_{54}} \Delta_2, \quad T_2 = \frac{1}{2\pi} \frac{\sqrt{2a_{54}}}{\Delta_2}, \quad (5.2)$$

where

$$\Delta_2 \equiv 2a_{54} + c^2.$$

Note that the extremal limit of black hole 1 (black hole 2) is obtained as  $a_{21} \rightarrow 0$  ( $a_{54} \rightarrow 0$ ), for  $b \neq 0$  ( $c \neq 0$ ). Note also that each of these black holes obeys a Smarr-type formula:

$$\frac{2}{3} M_i^{\text{Komar}} = T_i \frac{\mathcal{A}_i}{4} + \Omega_i^\phi J_i^\phi, \quad i = 1, 2. \quad (5.3)$$

## 5.2 Asymptotics and physical quantities

We now show that the solution is asymptotically flat and read off the ADM mass and angular momentum. Introducing the asymptotic coordinates  $r$  and  $\theta$

$$\rho = \frac{1}{2} r^2 \sin 2\theta, \quad z = \frac{1}{2} r^2 \cos 2\theta, \quad (5.4)$$

the asymptotic limit becomes  $r \rightarrow \infty$ . We can check that  $G_{tt} = -1 + O\left(\frac{1}{r^2}\right)$ , as expected, and fix  $q$  by requiring that  $G_{t\phi} \rightarrow 0$  as  $r \rightarrow \infty$ ; this yields

$$q = \frac{\tilde{b} + \tilde{c}}{a_{41}}. \quad (5.5)$$

For the conformal factor  $e^{2\nu(\rho,z)}$  we have, asymptotically,

$$e^{2\nu} = \frac{k}{r^2} + O\left(\frac{1}{r^4}\right), \quad (5.6)$$

which fixes  $k = 1$ . Thus, at infinity, the metric reduces to the standard form in bipolar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 + r^2 \cos^2 \theta d\phi^2, \quad (5.7)$$

so that taking the canonical periods  $\Delta\phi = 2\pi = \Delta\psi$  guarantees absence of conical singularities at infinity.

From the next to leading order term in  $G_{tt}$  and leading order term in  $G_{t\phi}$  we can read off the ADM mass and angular momentum to be

$$M_{\text{ADM}} = \frac{3\pi}{8} \left[ 2a_{21} + 2a_{54} + \frac{(\tilde{b} + \tilde{c})^2}{a_{41}^2} \right], \quad (5.8)$$

$$J_{\text{ADM}}^\phi = \frac{\pi}{4} \left[ \frac{2[\tilde{b}(a_{21} + a_{54} + a_{34}) + \tilde{c}(a_{21} + a_{54} + a_{31})]}{a_{41}} + \frac{(\tilde{b} + \tilde{c})^3}{a_{41}^3} \right]. \quad (5.9)$$

Note that these expressions i) are invariant under (4.13) as one would expect; ii) reduce to (5.1) and (5.2) in the limits  $a_3 = a_4 = a_5$  and  $a_1 = a_2 = a_3$ , respectively. Note also that the ordering (3.1) guarantees positivity of the ADM mass.

### 5.3 Conical singularities

The conical *excesses* for the generic solution are

$$\delta_\psi = 2\pi \left( \frac{a_{41}a_{52}}{\sqrt{a_{51}a_{31}a_{32}a_{42}}} \cdot \left| \frac{2a_{42}a_{51}}{2a_{42}a_{51} + b\tilde{c}} \right| - 1 \right), \quad a_2 \leq z < a_3; \quad (5.10)$$

$$\delta_\phi = 2\pi \left( \frac{a_{41}a_{52}}{\sqrt{a_{51}a_{43}a_{53}a_{42}}} \cdot \left| \frac{2a_{42}a_{51}}{2a_{42}a_{51} - \tilde{b}c} \right| - 1 \right), \quad a_3 < z \leq a_4. \quad (5.11)$$

These reduce to (3.4) when  $b = 0 = c$  and to (2.2) if also  $a_1 = a_2$  and  $a_4 = a_5$ . Note that the second condition should only be considered if one imposes the axis condition.

It is clear that the introduction of rotation could eliminate either of these conical singularities, but not *both* simultaneously. However, one must note that the requirement for either of these conical singularities to vanish is *incompatible* with the axis condition. To see this, require first  $\delta_\psi = 0$ . This demands  $bc > 0$ . We already know that the axis condition puts an upper bound on the positive values of  $bc$ ; thus, we can parametrise the possible values of  $bc$  as

$$bc = \frac{2a_{51}a_{42}}{a_{31}}\epsilon, \quad 0 \leq \epsilon \leq 1.$$

Substituting in (5.11) we observe that  $\epsilon \neq 1$  do avoid a divergence in  $\delta_\phi$ . The condition that  $\delta_\psi = 0$  becomes

$$\epsilon = \frac{a_{31}}{a_{43}} \left( \frac{a_{41}a_{52}}{\sqrt{a_{51}a_{31}a_{32}a_{42}}} - 1 \right).$$

It is fairly simple to show that the r.h.s. of this last equation is always greater or equal to 1; since we have seen that the l.h.s. is smaller than one we can conclude that  $\delta_\psi$  *cannot be set to zero and, at the same time, obey the axis condition*. Thus we can set  $\delta_\psi = 0$ , which regularises this conical singularity but, generically, the geometry will develop closed timelike curves.

Let us now require  $\delta_\phi = 0$ . This demands  $bc < 0$ , in fact

$$bc = \frac{2a_{42}a_{51}}{a_{31}}(1 - \beta), \quad \beta \equiv \frac{a_{41}a_{52}}{\sqrt{a_{51}a_{43}a_{53}a_{42}}}. \quad (5.12)$$



Note that  $\beta \geq 1$ . Replacing in (4.18) one gets

$$c^2 = \frac{2a_{42}^2 a_{51} \beta (1 - \beta)}{a_{41}^2 - a_{42} a_{43} \beta},$$

whose r.h.s. is manifestly negative (observe that  $a_{41}^2 \geq a_{42} a_{43} \beta$ ). Thus  $\delta_\phi$  cannot be set to zero and, at the same time, obey the axis condition. Notice therefore that  $\delta_\phi = 0$  cannot be interpreted as a regularity condition, since, when it is obeyed,  $\rho = 0$ ,  $a_3 < z < a_4$  is not an axis.

The incompatibility of the axis and regularity conditions is reminiscent of the result obtained in [15] for  $D = 4$  using a post-post Newtonian analysis.

### 5.4 Horizons geometry, areas and temperatures

Let us now show that both black holes have, in general, regular (except for a conical singularity at one point) finite area horizons and finite temperatures.

The horizon of the first black hole is located at  $\rho = 0$  and  $a_1 < z < a_2$ . Considering the coordinate transformation

$$\rho = \frac{1}{2} \sqrt{1 - \frac{2a_{21}}{R^2}} R^2 \sin 2\theta, \quad z = \frac{a_1 + a_2}{2} + \frac{1}{2} (R^2 - a_{21}) \cos 2\theta, \quad (5.13)$$

the horizon is located at  $R^2 = 2a_{21}$ . Note that  $z = a_2 - a_{21} \sin^2 \theta$ . The metric on a spatial section of the horizon reads

$$ds_{H_1}^2 = \frac{a_{31} a_{51}}{a_{41}^2} \Sigma(\theta) f_1(\theta) d\theta^2 + \frac{f_2(\theta) \cos^2 \theta}{\Sigma(\theta)} \Delta^2 d\phi^2 + 2a_{21} f_3(\theta) \sin^2 \theta d\psi^2, \quad (5.14)$$

where

$$\Sigma(\theta) \equiv F_1(\theta) \sin^2 \theta + 2a_{21} (1 + F_2(\theta))^2,$$

with the functions  $F(\theta)$  given by

$$F_1(\theta) \equiv b^2 \frac{a_{41}^2 f_2(\theta)}{a_{51}^2 f_1(\theta)}, \quad F_2(\theta) \equiv bc \frac{a_{43} \cos^2 \theta}{a_{51} 2(a_{42} + a_{21} \sin^2 \theta)},$$

and the functions  $f(\theta)$  given by

$$f_1(\theta) \equiv \frac{a_{42} + a_{21} \sin^2 \theta}{a_{52} + a_{21} \sin^2 \theta}, \quad f_2(\theta) \equiv \frac{a_{32} + a_{21} \sin^2 \theta}{a_{42} + a_{21} \sin^2 \theta}, \quad f_3(\theta) \equiv \frac{a_{52} + a_{21} \sin^2 \theta}{a_{32} + a_{21} \sin^2 \theta}. \quad (5.15)$$

The area and temperature of this black hole are given by

$$\mathcal{A}_1 = 2\pi^2 \frac{\sqrt{2a_{21} a_{31} a_{51}}}{a_{41}} \Delta, \quad T_1 = \frac{1}{2\pi} \sqrt{\frac{2a_{21}}{a_{31} a_{51}}} \frac{a_{41}}{\Delta}. \quad (5.16)$$

Note that the results (5.14)–(5.16) reduce to the expressions in [19], for  $b = 0$ , and to the ones of a single Myers-Perry black hole for  $a_3 = a_4 = a_5$ , in particular to (5.1).

A similar analysis can be done for the horizon of the second black hole, which is located at  $\rho = 0$  and  $a_4 < z < a_5$ . Considering the coordinate transformation

$$\rho = \frac{1}{2} \sqrt{1 - \frac{2a_{54}}{R^2}} R^2 \sin 2\theta, \quad z = \frac{a_4 + a_5}{2} + \frac{1}{2} (R^2 - a_{54}) \cos 2\theta, \quad (5.17)$$

the horizon is located at  $R^2 = 2a_{54}$ . Note that  $z = a_5 - a_{54} \sin^2 \theta$ . The metric on a spatial section of the horizon reads

$$ds_{H_2}^2 = \frac{a_{53}a_{51}}{a_{52}^2} \bar{\Sigma}(\theta) \bar{f}_1(\theta) d\theta^2 + \frac{\bar{f}_2(\theta) \cos^2 \theta}{\bar{\Sigma}(\theta)} \bar{\Delta}^2 d\phi^2 + 2a_{54} \bar{f}_3(\theta) \sin^2 \theta d\psi^2. \quad (5.18)$$

where

$$\bar{\Sigma}(\theta) \equiv \bar{F}_1(\theta) \sin^2 \theta + 2a_{54}(1 + \bar{F}_2(\theta)),$$

with the functions  $\bar{F}(\theta)$  given by

$$\bar{F}_1(\theta) \equiv \frac{\tilde{c}^2 \bar{f}_2(\theta)}{a_{41}^2 \bar{f}_1(\theta)}, \quad \bar{F}_2(\theta) = \frac{\tilde{b} a_{54} \sin^2 \theta \cos^2 \theta}{a_{41}^2 a_{51}^2 (a_{42} + a_{54} \cos^2 \theta)} \left[ a_{51} \tilde{c} \bar{f}_2(\theta) + \frac{\tilde{b} a_{54}^2 \cos^2 \theta}{2(a_{43} + a_{54} \cos^2 \theta)} \right],$$

and the functions  $\bar{f}(\theta)$  given by

$$\bar{f}_1(\theta) \equiv \frac{a_{42} + a_{54} \cos^2 \theta}{a_{41} + a_{54} \cos^2 \theta}, \quad \bar{f}_2(\theta) \equiv \frac{a_{41} + a_{54} \cos^2 \theta}{a_{43} + a_{54} \cos^2 \theta}, \quad \bar{f}_3(\theta) \equiv \frac{a_{43} + a_{54} \cos^2 \theta}{a_{42} + a_{54} \cos^2 \theta}. \quad (5.19)$$

The area and temperature of this black hole are given by

$$\mathcal{A}_2 = 2\pi^2 \frac{\sqrt{2a_{54}a_{53}a_{51}}}{a_{52}} \bar{\Delta}, \quad T_2 = \frac{1}{2\pi} \sqrt{\frac{2a_{54}}{a_{53}a_{51}}} \frac{a_{52}}{\bar{\Delta}}. \quad (5.20)$$

Note that the results (5.18)–(5.20) reduce to the expressions in [19], for  $b = 0 = c$ , and to the ones of a single Myers-Perry black hole for  $a_1 = a_2 = a_3$ , in particular to (5.2).

The surfaces described by (5.14) and (5.18) are topologically 3-spheres; they are regular, for generic parameters, except for a conical singularity at  $\theta = 0$  for the first black hole, where there is a conical excess in  $\psi$  given by (5.10), and at  $\theta = \pi/2$  for the second black hole, where there is a conical excess in  $\phi$  given by (5.11), *if one imposes* the axis condition (4.17). Note that at  $\theta = \pi/2$  for the first black hole, and at  $\theta = 0$  for the second, there are no conical singularities, in agreement with our discussion of section 2.

### 5.5 Individual masses and angular momenta

The individual mass of each black hole can be computed as a Komar integral at the horizon of each black hole. In five dimensions, and for a metric of type (4.10) the integral takes the form

$$M^{\text{Komar}} = \frac{3}{32\pi G_5} \int_S \star d\xi = \frac{3}{32\pi G_5} \int_{H_i} dz d\phi d\psi \frac{g_{\rho\rho} g_{\psi\psi}}{\sqrt{-g}} [g_{t\phi} g_{t\phi, \rho} - g_{\phi\phi} g_{tt, \rho}],$$

where  $\xi = g_{tt} dt + g_{t\phi} d\phi$  is the one-form dual to the asymptotic time translations Killing vector field  $\partial/\partial t$  and  $S$  is the boundary of any spacelike hypersurface; to derive the second equality we have already chosen  $S$  to be a spatial section of the event horizon of one of the two black holes. Thus  $a_1 < z < a_2$  ( $a_4 < z < a_5$ ) for the first (second) black hole. We find

$$M_1^{\text{Komar}} = \frac{3\pi}{8} \frac{2a_{42}a_{51}}{2a_{42}a_{51} + b\tilde{c}} \Delta, \quad M_2^{\text{Komar}} = \frac{3\pi}{8} \bar{\Delta}. \quad (5.21)$$

The intrinsic spin of each black hole can also be computed as a Komar integral at the horizon of each black hole. In five dimensions, and for a metric of type (4.10) the integral takes the form

$$J^{\text{Komar}} = -\frac{1}{16\pi G_5} \int_S \star d\zeta = -\frac{1}{16\pi G_5} \int_{H_i} dz d\phi d\psi \frac{g_{\rho\rho} g_{\psi\psi}}{\sqrt{-g}} [g_{t\phi} g_{\phi\phi,\rho} - g_{\phi\phi} g_{t\phi,\rho}] , \quad (5.22)$$

where  $\zeta = g_{\phi\phi} d\phi + g_{t\phi} dt$  is the one-form dual to the azimuthal Killing vector field  $\partial/\partial\phi$  and  $S$  is the boundary of any spacelike hypersurface; again, for the second equality we have already chosen  $S$  to be a spatial section of the event horizon of one of the two black holes. Thus  $a_1 < z < a_2$  ( $a_4 < z < a_5$ ) for the first (second) black hole. We find

$$J_1^{\text{Komar}} = \frac{\pi}{4} \frac{2a_{51}(a_{42}\tilde{b} - a_{21}\tilde{c})}{a_{41}(2a_{42}a_{51} + b\tilde{c})} \Delta , \quad J_2^{\text{Komar}} = \frac{\pi}{4} \frac{(\tilde{b} + \tilde{c})}{a_{41}} \bar{\Delta} . \quad (5.23)$$

Thus the angular momentum to mass ratio of any of the individual black holes has a very simple expression

$$j_1 \equiv \frac{J_1^{\text{Komar}}}{M_1^{\text{Komar}}} = \frac{2}{3a_{41}} \left( \tilde{b} - \frac{a_{21}}{a_{42}} \tilde{c} \right) , \quad j_2 \equiv \frac{J_2^{\text{Komar}}}{M_2^{\text{Komar}}} = \frac{2}{3a_{41}} (\tilde{b} + \tilde{c}) .$$

A simple interpretation for the parameter  $c$  follows: it is, up to a constant, the *difference in angular momentum per unit mass of the two black holes*

$$c = \frac{3}{2} \frac{a_{42}}{a_{43}} (j_2 - j_1) . \quad (5.24)$$

The parameter  $b$ , on the other hand, is a measure of the *sum* of the angular momentum per unit mass of the two black holes since

$$b = \frac{3}{2} \left( \frac{a_{42}}{a_{31}} j_1 + \frac{a_{21}}{a_{31}} j_2 \right) . \quad (5.25)$$

Note that, if  $c = 0$ ,

$$j_1 = j_2 = \frac{2}{3} \frac{a_{31}}{a_{41}} b . \quad (5.26)$$

Thus, one should regard  $b$  as turning on the angular momentum per unit mass of both black holes, and one should think of  $c$  as turning on the difference in angular momentum per unit mass of the two black holes.

One can turn off the intrinsic spin of either black hole by imposing the conditions

$$j_1 = 0 \Leftrightarrow \tilde{b} = \frac{a_{21}}{a_{42}} \tilde{c} , \quad j_2 = 0 \Leftrightarrow \tilde{b} = -\tilde{c} .$$

One can, however, show that neither these conditions is compatible with the axis condition (4.17) and non-trivial  $b$  and  $c$ . This is most easily done re-expressing the axis condition in terms of  $j_1$  and  $j_2$ . We get

$$\Delta_{\text{axis}} = \frac{3}{2} \frac{a_{41} a_{42}}{a_{43}} \left[ 2a_{51}(a_{41}j_1 - a_{31}j_2) - \frac{9}{4} j_2(j_2 - j_1)(a_{42}j_1 + a_{21}j_2) \right] . \quad (5.27)$$

The Komar masses and angular momenta, (5.21) and (5.22), obey, together with the temperatures and areas (5.16) and (5.20), Smarr relations (5.3), as in the static case and, for instance, the black saturn solution; but unlike these backgrounds, for our solution the Komar masses and angular momenta, in general, do not add up to the ADM mass and angular momentum, since

$$M_{\text{ADM}} = M_1^{\text{Komar}} + M_2^{\text{Komar}} + M_{\text{extra}}^{\text{Komar}} , \quad (5.28)$$

$$J_{\text{ADM}} = J_1^{\text{Komar}} + J_2^{\text{Komar}} + J_{\text{extra}}^{\text{Komar}} . \quad (5.29)$$

The reason is that, in general, there is a non-trivial Komar integral coming from the surface  $S_\phi$ , which is given by  $\rho = 0$ ,  $a_3 < z < a_4$ . This contribution is only present if the axis condition is not obeyed and it accounts for the extra piece in the last two equations:

$$\begin{aligned} M_{\text{extra}}^{\text{Komar}} &= \frac{3}{32\pi G_5} \int_{S_\phi} dz d\phi d\psi \frac{g_{\rho\rho} g_{\psi\psi}}{\sqrt{-g}} [g_{t\phi} g_{t\phi,\rho} - g_{\phi\phi} g_{t\rho}] \\ &= -\frac{3\pi}{8} \frac{a_{43} b \Delta_{\text{axis}}}{a_{41} a_{51} (2a_{42} a_{51} + b\tilde{c})} , \end{aligned} \quad (5.30)$$

$$J_{\text{extra}}^{\text{Komar}} = -\frac{1}{16\pi G_5} \int_{S_\phi} dz d\phi d\psi \frac{g_{\rho\rho} g_{\psi\psi}}{\sqrt{-g}} [g_{t\phi} g_{\phi\phi,\rho} - g_{\phi\phi} g_{t\rho}] \quad (5.31)$$

$$= -\frac{a_{43} \Delta_{\text{axis}}}{3a_{51} a_{41}^2 a_{42}} \left( \frac{3\pi}{4} a_{42} + M_1^{\text{Komar}} \right) . \quad (5.32)$$

Note that the extra piece is indeed proportional to  $a_{43}$ . Imposing the axis condition, the Komar masses and angular momenta do add up to the ADM mass and angular momentum.

## 6. Discussion and conclusions

In this paper we have used the inverse scattering technique to generate a new asymptotically flat, vacuum solution of five dimensional general relativity describing two Myers-Perry black holes, each with a singular angular momentum parameter, both in the same plane. We have described the basic properties and physical quantities of the solution as well as of the background geometry it is built upon. In general the solution has conical singularities in both spatial 2-planes. The conical singularity in the  $\rho - \psi$  plane can be removed if

$$b\tilde{c} = 2a_{42} a_{51} \left( \frac{a_{41} a_{52}}{\sqrt{a_{51} a_{31} a_{32} a_{42}}} - 1 \right) .$$

On the other hand, the conical singularity in the  $\phi$  plane cannot be removed. Indeed, when the axis condition is imposed, which guarantees that  $\rho = 0$ ,  $a_3 < z < a_4$  is an axis,  $\delta_\phi \neq 0$ . The axis condition, which has been interpreted as a torque balance condition is

$$(\tilde{b} + \tilde{c})(2a_{42} a_{51} - \tilde{b}c) = 2a_{41}^2 a_{51} c .$$

It would be interesting to have a physical interpretation of these conditions in terms of the different forces and torques that play a role in this geometry. This might be possible to do using an energetics analysis along the lines of [18, 45], a problem we expect to address in the future.

One somewhat unexpected feature that we found was a contribution to the ADM mass and angular momentum of one part of the geometry exterior to the black hole horizons, if the axis condition is not obeyed. This suggests that, in the post-post Newtonian analysis of this type of problems, along the lines of [12, 15], one should indeed include one further parameter describing the rotating rod, as suggested in [46]. This might clarify the discrepancy between the result obtained in the post-post Newtonian analysis and the one obtained from the exact double Kerr solution, for the regularity and axis conditions in the case of two massive spinning particles in  $D = 4$ .

Finally let us remark that in the five dimensional family of supersymmetric multi-black hole spacetimes known as BMPV [47, 48], no condition is required, analogous to the axis condition that has to be imposed for the IWP spacetimes. This is in curious contrast with the smoothness properties of horizons in static multi-centre solutions, pointed out in [49, 50], which get worse in five than in four dimensions and still worse in higher dimensions.

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